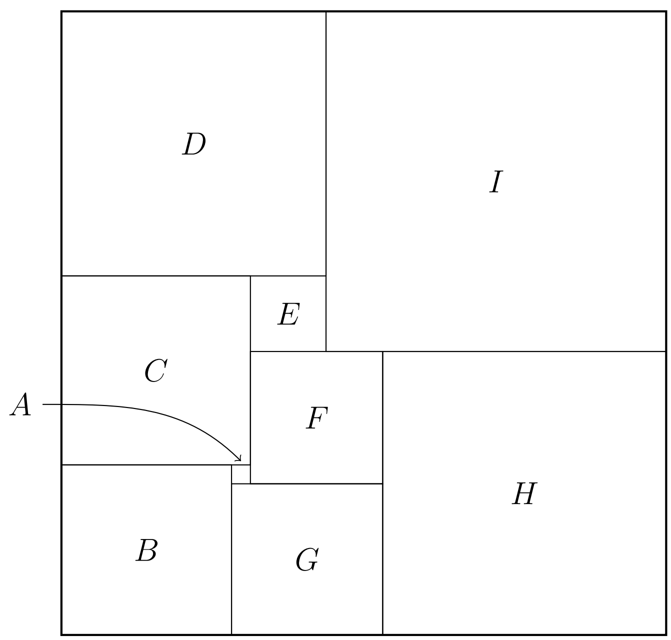
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 15, September 3, 2018

1. In a plane lie 127 cogs. The teeth of Cog-1 engage those of Cog-2. The teeth of Cog-2 engage those of Cog-3 and so on. Finally the teeth of Cog-127 engage those of Cog-1.

   Can the cog wheels so arranged be turned?

2. Nine squares are arranged to form a rectangle as shown. If the smallest square has area one, find the total area of the rectangle.

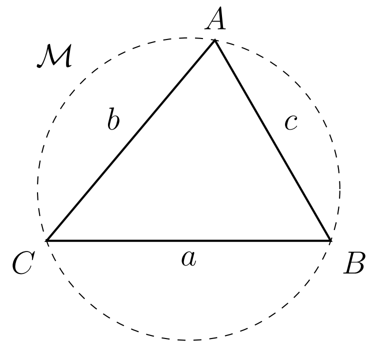


3. **Construction problem:** Construct a parallelogram, given its sides, the sum of the diagonals, and the angle between them.[1]

4. A powerful number is an integer whose prime factors, when squared, remain factors. A perfect power is an integer which can be written as another integer to an integer power. Find the smallest positive integer which is powerful but not a perfect power.

[1] Adapted from AP Kiselev *Kiselev's Geometry: Planimetry*, Tr. A Givental, 2006

5. Suppose that ABC is a triangle with circumcircle, $\mathcal{M}$, as shown below.



   (a) Show that $\dfrac{a}{\sin A} = 2r$, where r is the radius of $\mathcal{M}$.

   (b) Hence derive the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Senior Questions

1. Consider an equilateral triangular hole, and the piece that fits into it. The *symmetry group of an equilateral triangle* is made up of the operations you can do to the piece so that it still fits in its hole. For instance, you can rotate it by 60°.

   (a) There are 6 operations in total: list them.

   (b) By labelling the corners of the triangle, show that these operations don't necessarily commute. That is, if you perform operation 1 first then operation 2, the outcome is not necessarily the same as when operation 2 is followed by operation 1.

   (c) There is an operation, e, called the identity, such that if x is any other operation $e \cdot x = x \cdot e = x$. Which operation does e correspond to?

   (d) Each operation has an "undo" operation called its inverse, such that if x is an operation and e is the identity operation, there's a y such that $x \cdot y = y \cdot x = e$. For each of the operations in the symmetry group of an equilateral triangle, list the inverse. Show also that for each operation x, its inverse is unique. That is, there's only one operation y such that $x \cdot y = y \cdot x = e$.